Dynamic Instability of Shear-Deformable Laminated Plates

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The dynamic instability of antisymmetric angle-ply and cross-ply laminated plates, subjected to periodic in-plane loads $P(t) = P_s + P_d \cos\omega t$, is investigated within the high-order shear deformation lamination theory, where the motion is governed by five partial differential equations. Using the method of multiple-scale, analytical expressions for the instability regions are obtained at $\omega = \Omega_j \pm \Omega_i$, where Ω_i are the natural frequencies of the system. It is shown that, besides the principal instability region at $\omega = 2\Omega_1$ (Ω_1 is the fundamental frequency), other cases of $\omega = \Omega_i + \Omega_j$, related to the first mode, can be of major importance and yield a significantly enlarged instability region.

Introduction

IN the analysis of the dynamic stability of a structure, subjected to a periodic loading $P(t) = P_s + P_d \cos\omega t$, it turns out that for certain relationships between the driving frequency and the natural ones, dynamic instability occurs, in the sense that the amplitude of the response increases without bound. The problem of the dynamic instability of elastic structures (columns, plates, and shells) was investigated by Bolotin,[1] where the instability regions were constructed by using the Fourier analysis. Extensive bibliography and further results on this problem were given by Evan-Iwanoswski[2,3] in a review paper and in a monograph.

The intensive use of fiber-reinforced composites has resulted lately in several studies of the dynamic instability of laminated plates. Birman[4] considered the unsymmetric laminates, using the classical lamination theory (CPT), neglecting rotary inertia. Srinivasan and Chellapandi[5] used the same theory and the finite strip method to determine the instability regions. The use of refined theories for the investigation of the dynamic instability of laminated plates was carried out by Bert and Birman[6] for angle-ply laminates and by Librescu and Thangjitham[7] for cross-ply laminates. In all these studies the so-called principal instability region, at which $\omega = 2\Omega_1$ (Ω_1 is the lowest natural frequency of the plate), was derived, and the effects of the aspect ratio, static in-plane force, and the number of layers were investigated.

However, since the motion of laminated plates is governed by several equations (according to the number of degrees of freedom), instability may occur at $\omega = \Omega_j \pm \Omega_i$, where $i,j = 1,2,\ldots,m$ and m is the number of degrees of freedom, as was already shown in Refs. 8–10, and the case of $\omega = 2\Omega_1$ is but a particular one.

In a previous paper by Mond and Cederbaum,[11] the principal instability regions of antisymmetric angle-ply and cross-ply laminates, using the method of multiple-scale, were investigated. The laminates were modeled by the CPT so that their motion was governed by three second-order differential equations. Analytical expressions for the various instability regions were obtained, and it was shown that the instability region is highly enlarged when combination resonances are taken into account.

In this paper we investigate the dynamic instability of laminated plates, modeled within the high-order shear deformation theory (HSDT), developed by Reddy.[12] The first aim of this paper is to determine analytical expressions for the possible instability regions within the HSDT.

As has already been shown (e.g., Refs. 13, 14), the fundamental natural frequencies determined by using the HSDT are lower than those obtained within the CPT. This implies that instability regions of simple parametric resonance ($\omega = 2\Omega_i$) will be obtained at lower excitation frequencies. Yet, since the HSDT assumes two more degrees of freedom, the number of possible combination resonance ($\omega = \Omega_j \pm \Omega_i$) is highly enlarged. The second aim of this paper is to investigate the contribution of these instability regions, obtained for the first mode, to the general instability region.

In this paper we do not include the effect of damping on the derived stability boundaries. The importance of the damping in the analysis of the dynamic stability of laminated plates is shown in Ref. 15.

Problem Formulation

Consider a rectangular ($a \times b$) laminated plate, subjected to uniformly distributed parametric excitations, as shown in Fig. 1. Within the HSDT, the equations of motion of such a plate are

$$
\begin{aligned}
&N_{1,x} + N_{6,y} = I_1\ddot{U} + (I_2 - cI_4)\ddot{\psi}_x - cI_4\ddot{W}_{,x} \\
&N_{6,x} + N_{2,y} = I_2\ddot{V} + (I_2 - cI_4)\ddot{\psi}_y - cI_4\ddot{W}_{,y} \\
&Q_{5,x} + Q_{4,y} + c(P_{1,xx} + 2P_{6,xy} + P_{2,yy}) \\
&\qquad - 3c(K_{5,x} + K_{4,y}) + N_x(t)W_{,xx} + N_y(t)W_{,yy} \\
&\qquad = I_1\ddot{W} - cI_7(\ddot{W}_{,xx} + \ddot{W}_{,yy}) + cI_4(\ddot{U}_{,x} + \ddot{V}_{,y}) \\
&\qquad + (cI_5 - c^2I_7)(\ddot{\psi}_{x,x} + \ddot{\psi}_{y,y}) \\
&M_{1,x} + M_{6,y} - c(P_{1,x} + P_{6,y}) - Q_5 + 3cK_5 \\
&\qquad = (I_2 - cI_4)\ddot{U} + (I_3 - 2cI_5 + c^2I_7)\ddot{\psi}_x \\
&\qquad - (cI_5 - c^2I_7)\ddot{W}_{,x} \\
&M_{6,x} + M_{2,y} - c(P_{6,x} + P_{2,y}) - Q_4 + 3cR_4 \\
&\qquad = (I_2 - cI_4)\ddot{V} + (I_3 - 2cI_5 + c^2I_7)\ddot{\psi}_y \\
&\qquad - (cI_5 - c^2I_7)\ddot{W}_{,y}
\end{aligned}
\tag{1}
$$

where U, V, and W denote the displacements of a point (x,y) on the mid-plane, $z = 0$, ψ_x, and ψ_y are the rotations of the normals to the mid-plane about the y and x axes, respectively, $c = 4h^2/3$, and $(\ )_{,i}$ and $(\dot{\ })$ denote differentiation with respect

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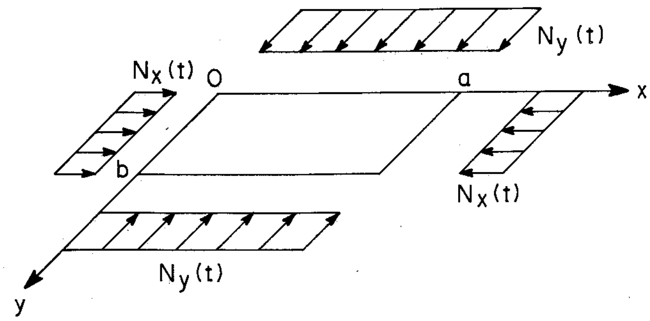

Fig. 1 Dimensional scheme of a plate subjected to in-plane loading.

to x or y and with time, respectively. The in-plane loadings are given by

$$N_x(t) = N_{xs} + N_{xd}\cos\omega t$$
$$N_y(t) = N_{ys} + N_{yd}\cos\omega t \qquad (2)$$

where N_{xs}, N_{xd}, N_{ys}, and N_{yd} are constants, ω is the load frequency, and t is time. The resultants are related to the generalized displacements by the following constitutive relations

$$\begin{Bmatrix} \{N_i\} \\ \{M_i\} \\ \{P_i\} \end{Bmatrix} = \begin{bmatrix} [A_{ij}] & [B_{ij}] & [E_{ij}] \\ & [D_{ij}] & [F_{ij}] \\ \text{symmetric} & & [H_{ij}] \end{bmatrix} \begin{Bmatrix} \{\varepsilon_j\} \\ \{k_j\} \\ \{\bar{k}_j\} \end{Bmatrix} \quad i,j = 1,2,6$$

$$\begin{Bmatrix} \{Q_j\} \\ \{K_j\} \end{Bmatrix} = \begin{bmatrix} [A_{ji}] & [D_{ji}] \\ \text{symmetric} & [F_{ji}] \end{bmatrix} \begin{Bmatrix} \varepsilon_i \\ \bar{k}_i \end{Bmatrix} \quad j,i = 4,5 \qquad (3)$$

where

$$\varepsilon_1 = U_{,x}; \quad \varepsilon_2 = V_{,y}; \quad \varepsilon_4 = \psi_y + W_{,y}$$
$$\varepsilon_5 = \psi_x + W_{,x}; \quad \varepsilon_6 = U_{,y} + V_{,x}$$
$$k_1 = \psi_{x,x}; \quad k_2 = \psi_{y,y}; \quad k_6 = \psi_{x,y} + \psi_{y,x}$$
$$\bar{k}_1 = -c(\psi_{x,x} + W_{,xx}); \quad \bar{k}_2 = -c(\psi_{y,y} + W_{,yy})$$
$$\bar{k}_4 = -3c(\psi_y + W_{,y}); \quad \bar{k}_5 = -3c(\psi_x + W_{,x})$$
$$\bar{k}_6 = -c(\psi_{x,y} + \psi_{y,x} + 2W_{,xy})$$

$$(A_{ij}, B_{ij}, D_{ij}, E_{ij}, F_{ij}, H_{ij}) = \int_{-h/2}^{h/2} \bar{Q}_{ij}(1, z, z^2, z^3, z^4, z^6)\,dz \quad (i,j = 1,2,6)$$

$$(A_{ji}, D_{ji}, F_{ji}) = \int_{-h/2}^{h/2} \bar{Q}_{ji}(1, z^2, z^4,)\,dz \quad (j,i = 4,5)$$

h is the plate thickness and $\bar{Q}_{ij}$ are the reduced elastic constants.

The inertia terms are given by

$$(I_1, I_2, I_3, I_4, I_5, I_7) = \int_{-h/2}^{h/2} \rho(1, z, z^2, z^3, z^4, z^6)\,dz$$

where ρ is the mass density.

Consider first the case of antisymmetric angle-ply laminate, hinged on all edges and the tangential stresses and in-plane displacements in the direction perpendicular to each edge are zero. For this laminate

$$A_{16} = A_{26} = B_{11} = B_{12} = B_{22} = B_{66} = D_{16} = D_{26} = 0$$
$$F_{16} = F_{26} = E_{11} = E_{12} = E_{22} = E_{66} = H_{16} = H_{26} = 0 \qquad (4)$$
$$A_{45} = D_{45} = F_{45} = 0$$

and the boundary conditions are

$$W = M_1 = U = N_6 = P_1 = \psi_y = 0 \quad \text{at} \quad x = 0,a$$
$$W = M_2 = V = N_6 = P_2 = \psi_x = 0 \quad \text{at} \quad y = 0,b \qquad (5)$$

These boundary conditions are satisfied by the following displacements field, when using the method of separation of variables,

$$U = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} u_{mn}(t)\sin\alpha x\cos\beta y$$
$$V = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} v_{mn}(t)\cos\alpha x\sin\beta y$$
$$W = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} w_{mn}(t)\sin\alpha x\sin\beta y$$
$$\psi_x = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} x_{mn}(t)\cos\alpha x\sin\beta y$$
$$\psi_y = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} y_{mn}(t)\sin\alpha x\cos\beta y \qquad (6)$$

where

$$\alpha = m\pi/a, \qquad \beta = n\pi/b$$

Substituting Eqs. (2), (3), and (6) into Eq. (1) we obtain a set of second-order differential equations which governs the motion of the antisymmetric angle-ply laminated plate subjected to in-plane parametric loading. This set of equations can be presented as

$$[T']\{Z\} - \varepsilon\cos\omega t[S']\{Z\} = [G]\{\ddot{Z}\} \qquad (7)$$

where $\{Z\}^T = \{u, v, w, x, y\}$, $[S']$ is a zero matrix except of $S'_{33} = N$, $\varepsilon = (N_{xd}\alpha^2 + N_{yd}\beta^2)/N$, where N is the static buckling load, and the terms of the symmetric matrices T' and G are given in the Appendix of Ref. 13.

Antisymmetric cross-ply laminates are considered next. In this case

$$A_{16} = A_{26} = B_{16} = B_{26} = B_{66} = D_{16} = D_{26} = 0$$
$$F_{16} = F_{26} = E_{16} = E_{26} = E_{66} = H_{16} = H_{26} = 0 \qquad (8)$$
$$A_{45} = D_{45} = F_{45} = 0$$

The boundary conditions

$$W = M_1 = V = N_1 = P_1 = \psi_y = 0 \quad \text{at} \quad x = 0,a$$
$$W = M_2 = U = N_2 = P_2 = \psi_x = 0 \quad \text{at} \quad y = 0,b \qquad (9)$$

are satisfied by the following displacement field

$$U = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} u_{mn}(t)\cos\alpha x\sin\beta y$$
$$V = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} v_{mn}(t)\sin\alpha x\cos\beta y$$
$$W = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} w_{mn}(t)\sin\alpha x\sin\beta y \qquad (10)$$
$$\psi_x = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} x_{mn}(t)\cos\alpha x\sin\beta y$$
$$\psi_y = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} y_{mn}(t)\sin\alpha x\cos\beta y$$

Using the same steps as in the previous case, one obtains Eq. (7) again where T'_{ij} and G_{ij} are also given in the appendix of Ref. 13.

Stability Analysis

In this section the stability properties of a system whose behavior is governed by Eq. (7) is investigated. For this purpose the following transformation is performed

$$R = L^T Z \tag{11}$$

where L is a 5×5 matrix whose columns are the eigenvectors of T'.

Inserting transformation (11) into Eq. (7) and multiplying both sides by L^T results in the following equation for R:

$$\ddot{R} + (T - \varepsilon \cos\omega t\, S)R = 0 \tag{12}$$

where

$$T = L^T G^{-1} T' L \underline{=} [\text{diag } \Omega_i^2] \tag{13}$$

and

$$S = L^T G^{-1} S' L \tag{14}$$

In order to facilitate the discussion below, Eq. (12) is written componentwise:

$$\ddot{R}_i + \Omega_i^2 R_i - \varepsilon \cos\omega t \sum_{j=1}^{5} S_{ij} R_j = 0$$

$$i,j = 1, 2, 3, 4, 5 \tag{15}$$

The last term on the left-hand side (which actually is the sum of five terms) represents the coupling of the dynamical load to the normal modes of the system. It is the presence of these coupling terms which prevents obtaining exact analytical solutions for Eq. (15). However, since the amplitude of the dynamical load, ε, is assumed to be small, a perturbation scheme can be introduced in order to investigate its effect on the stability of the plate.

A regular perturbation scheme provides an adequate description of the systems in those regions in parameter space where the latter is stable. However, when the system approaches its stability boundaries, the regular perturbation scheme breaks down and the multiscale perturbation analysis must be employed.[16,17]

The multiscales analysis is based on the observation that the system described by Eq. (15) varies on two distinct time scales. The first time scale is given by the natural frequencies of the system, Ω_i, $i = 1, 2, 3, 4, 5$ defined to be of order 1. The second time scale, associated with the dynamical loading amplitude, is given by $1/\varepsilon$ and is much longer than the natural periods.

These two time scales that govern the behavior of the system give rise to replacing the independent variable t by two independent variables t and $\tau = \varepsilon t$. The latter represents the slow response of the system to the small dynamical loading amplitude. As a result, the time derivatives in Eq. (15) are replaced by the following partial derivatives:

$$\frac{dR}{dt} = \frac{\partial R}{\partial t} + \varepsilon \frac{\partial R}{\partial \tau} \tag{16}$$

In addition, R is expanded in power series in ε in the following way:

$$R = R_0(t,\tau) + \varepsilon R_1(t,\tau) + \cdots \tag{17}$$

As has already been mentioned, instability may occur at $\omega = \Omega_i + \Omega_j$ and at $\omega = \Omega_i - \Omega_j$. In the following sections we will investigate the cases of a) $\omega = 2\Omega_1$ and b) $\omega = \Omega_5 \pm \Omega_4$, representing the other possibilities of instability regions obtained for other Ω_i and Ω_j.

ω is Close to $2\Omega_1$

In this region Ω_1 is related to ω in the following way

$$\Omega_1^2 = \frac{\omega^2}{4} + \varepsilon\Omega_{11}^2 + O(\varepsilon^2) \tag{18}$$

Expansions (16–18) are inserted into Eq. (15) and coefficients of equal powers of ε are collected. The lowest order terms yield the following equations for R_0:

$$\frac{\partial^2 R_{10}}{\partial t^2} + \frac{\omega^2}{4} R_{10} = 0 \tag{19}$$

$$\frac{\partial^2 R_{j0}}{\partial t^2} + \Omega_j^2 R_{j0} = 0, \qquad j = 2, 3, 4, 5 \tag{20}$$

The solutions of Eqs. (19) and (20) are given by

$$R_{10}(t,\tau) = A_1(\tau)e^{i\frac{\omega}{2}t} + CC \tag{21}$$

$$R_{j0}(t,\tau) = A_j(\tau)e^{i\Omega_j t} + CC \qquad j = 2, 3, 4, 5 \tag{22}$$

where CC stands for complex conjugate and $A_j(t)$, $j = 1, 2, 3, 4, 5$ are functions to be determined.

Next, terms of first order in ε yield the following equations:

$$\frac{\partial^2 R_{11}}{\partial t^2} + \frac{\omega^2}{4} R_{11} = \left(-i\omega \frac{dA_1}{d\tau} - \Omega_{11}^2 A_1 + \frac{1}{2} S_{11} A_1^*\right) e^{i\frac{\omega}{2}t} + NST + CC \tag{23}$$

$$\frac{\partial^2 R_{j1}}{\partial t^2} + \Omega_j^2 R_{j1} = -2i\Omega_j \frac{dA_j}{d\tau} e^{i\Omega_j t} + NST + CC \qquad j = 2, 3, 4, 5 \tag{24}$$

The terms on the right-hand side of Eqs. (23) and (24) that multiply $\exp(i\omega t/2)$ and $\exp(i\Omega_j t)$, respectively, give rise to unbounded solutions. They are known in the literature as secular terms. The rest are nonsecular terms (NST). The secular terms are responsible for the breaking down of the regular perturbation scheme. Here, with the aid of the multiscale technique, the secular terms can be eliminated by picking the slowly varying amplitudes, A_j, $j = 1, 2, 3, 4, 5$, such that the coefficients of $\exp(i\omega t/2)$ and $\exp(i\Omega_j t)$, $j = 2, 3, 4, 5$ vanish. This requirement yields the following differential equations for the slowly varying amplitudes:

$$\omega \frac{dA_1}{d\tau} - i\Omega_{11}^2 A_1 + \frac{i}{2} S_{11} A_1^* = 0 \tag{25}$$

$$\frac{dA_j}{d\tau} = 0 \qquad j = 2, 3, 4, 5 \tag{26}$$

The solutions of Eqs. (25) and (26) are

$$A_1 = \alpha e^{(1/\omega)\sqrt{(1/4)S_{11}^2 - \Omega_{11}^4}\,\tau} + \beta^{-(1/\omega)\sqrt{(1/4)S_{11}^2 - \Omega_{11}^4}\,\tau} \tag{27}$$

$$A_j = \gamma_j \qquad j = 2, 3, 4, 5 \tag{28}$$

where α, β, and γ_j are constants.

From Eq. (27) it is clear that instability occurs when

$$\tfrac{1}{2}|S_{11}| > \Omega_{11}^2 \qquad (29)$$

Hence, with Eq. (18), the stability boundaries in the ω-ε plane are given by:

$$\Omega^2 = \frac{\omega^2}{4} \pm \frac{\varepsilon}{2} S_{11} = \frac{\omega^2}{4}\left[1 \pm \frac{2S_{11}}{\omega^2}\varepsilon\right] \qquad (30)$$

Similar stability regions can be found in the same fashion for the other four natural frequencies (Ω_2, Ω_3, Ω_4, and Ω_5). Yet, since Ω_1 is the lowest, the instability region given by Eq. (30) is the most important one.

ω is Close to ($\Omega_5 - \Omega_4$)

In this region Ω_4 and Ω_5 are expanded in power series in ε:

$$\Omega_j^2 = \Omega_{j0}^2 + \varepsilon\Omega_{j1}^2 + O(\varepsilon^2) \qquad j = 4, 5 \qquad (31)$$

such that

$$\Omega_{50} - \Omega_{40} = \omega \qquad (32)$$

As a result, the deviation of ω from ($\Omega_5 - \Omega_4$) is given by

$$\varepsilon\Delta = (\Omega_5 - \Omega_4) - \omega = \frac{\varepsilon}{2}\left(\frac{\Omega_{51}^2}{\Omega_{50}} - \frac{\Omega_{41}^2}{\Omega_{40}}\right) \qquad (33)$$

The quantity Δ is the mismatch between the frequency of the dynamical load and the difference between the two natural frequencies, Ω_5 and Ω_4.

Expansions (16), (17), and (31) are inserted into Eq. (15) and, again, coefficients of equal powers of ε are collected. The lowest order terms result in the following equations:

$$\frac{\partial^2 R_{i0}}{\partial t^2} + \Omega_i^2 R_{i0} = 0 \qquad i = 1, 2, 3 \qquad (34)$$

$$\frac{\partial^2 R_{j0}}{\partial t^2} + \Omega_{j0}^2 R_{j0} = 0 \qquad j = 4, 5 \qquad (35)$$

whose solutions are given by

$$R_{i0} = A_i(\tau)e^{i\Omega_i t} + CC \qquad i = 1, 2, 3 \qquad (36)$$

$$R_{j0} = A_j(\tau)e^{i\Omega_{j0} t} + CC \qquad j = 4, 5 \qquad (37)$$

Collecting terms of first order in ε and separating the secular from the nonsecular terms yields the following equations:

$$\frac{\partial^2 R_{i1}}{\partial t^2} + \Omega_i^2 R_{i1} = -2i\Omega_i \frac{dA_i}{d\tau} e^{i\Omega_i t} + NST + CC \qquad i = 1, 2, 3 \qquad (38)$$

$$\frac{\partial^2 R_{j1}}{\partial t^2} + \Omega_{j0}^2 R_{j1} = \left(-2i\Omega_{j0}\frac{dA_j}{d\tau} - \Omega_{j1}^2 A_j\right)e^{i\Omega_{j0}t} + \frac{1}{2} S_{jk} A_k e^{i[\Omega_{k0} + (j-k)\omega]t} + NST + CC \qquad (39)$$

where $j = 4, 5$ and $k = 5, 4$, respectively. It is easy to see that, in addition to the two terms (in brackets) on the right-hand side of Eq. (39), the third term is also secular when $\Omega_{50} - \Omega_{40} = \omega$.

The condition that no secular terms exist on the right-hand side of Eqs. (38) and (39) results in the following equations for the amplitudes:

$$\frac{dA_i}{d\tau} = 0 \qquad i = 1, 2, 3 \qquad (40)$$

$$\frac{dA_4}{d\tau} = \frac{i}{2}\frac{\Omega_{41}^2}{\Omega_{40}} A_4 - \frac{i}{2}\frac{S_{45}}{2\Omega_{40}} A_5 \qquad (41)$$

$$\frac{dA_5}{d\tau} = \frac{i}{2}\frac{\Omega_{51}^2}{\Omega_{50}} A_5 - \frac{i}{2}\frac{S_{54}}{2\Omega_{50}} A_4 \qquad (42)$$

Equation (40) means that R_{i0} vary only on the fast time scale, while the coupled Eqs. (41) and (42) are solved by

$$A_j = \alpha_j e^{\lambda\tau} \qquad j = 4, 5 \qquad (43)$$

where

$$\lambda = \frac{i}{4}\left(\frac{\Omega_{41}^2}{\Omega_{40}} + \frac{\Omega_{51}^2}{\Omega_{50}}\right) \pm \frac{1}{2}\sqrt{-\frac{S_{45}S_{54}}{4\Omega_{40}\Omega_{50}} - \Delta^2} \qquad (44)$$

Instability occurs when λ has a positive real part. However, since S is a symmetric matrix, λ is imaginary and hence the system is stable when the loading frequency approaches $\Omega_5 - \Omega_4$. Obviously, the same conclusion holds for any combination $\Omega_i - \Omega_j$.

ω is close to ($\Omega_4 + \Omega_5$)

In this case, the same steps are followed as those that led to Eq. (44). The result for the current case is obtained from Eq. (44) by replacing Ω_{50} by $-\Omega_{50}$. Thus, λ is given by

$$\lambda = \frac{i}{4}\left(\frac{\Omega_{41}^2}{\Omega_{40}} - \frac{\Omega_{51}^2}{\Omega_{50}}\right) \pm \frac{1}{2}\sqrt{\frac{S_{45}S_{54}}{4\Omega_{40}\Omega_{50}} - \Delta^2} \qquad (45)$$

Contrary to the previous case, now instability may occur. From Eq. (45) it is easy to see that this happens when

$$\frac{S_{45}S_{54}}{4\Omega_{40}\Omega_{50}} > \Delta^2 \qquad (46)$$

and the stability boundaries are given by

$$\Omega_4 + \Omega_5 = \omega \pm \frac{\varepsilon}{2}\sqrt{\frac{S_{45}S_{54}}{\Omega_{40}\Omega_{50}}} \qquad (47)$$

Again the same can be said for the other combinations of $\omega = \Omega_i + \Omega_j$, thus, the general expression for the stability boundaries for any i,j is

$$\Omega_i + \Omega_j = \omega \pm \frac{\varepsilon}{2}\sqrt{\frac{S_{ij}S_{ji}}{\Omega_{i0}\Omega_{j0}}} \qquad (48)$$

Results and Discussion

Equations (30) and (48) describe the various instability regions for the first mode, where $m = n = 1$ in Eq. (6). Other instability regions, for any other mode, can be formulated similarly by using the appropriate m and n. Our focus, however, will be given in the following to the first mode since it yields the most important instability region at $\omega = 2\Omega_1$, in the sense that this one is the widest. Yet, the total instability region might be significantly enlarged when the combination resonances are taken into account.

The following material properties

$$E_1 = 20E_2,\ G_{23} = 0.5E_2,\ G_{12} = G_{13} = 0.6E_2,\ \nu_{12} = 0.25$$

were considered for the square ($a \times a$) symmetric [0 deg, 90 deg, 90 deg, 0 deg] cross-ply laminates and the antisymmetric [θ, $-\theta$, θ, $-\theta$] angle-ply laminates.

Figure 2 shows the instability regions for a cross-ply laminate where $a/h = 5$. The main region, the principal one, is for $\omega = 2\Omega_1$ and where $S_{11}/2\Omega_1$ is normalized to unity, whereas the second one relates to $\omega = \Omega_1 + \Omega_2$. It can be easily seen that the second region contributes a considerable amount to the total instability region. The other possible combination resonances were found to be less important.

The importance of the various regions of instability can be inferred from Eqs. (30) and (48), where the regions' boundaries are straight lines, and the ratio of their slope to the slope of the principal one is given by

$$\bar{S}_{ij} = \frac{S_{ij}}{S_{11}} \left[\frac{\Omega_{10}^2}{\Omega_{i0}\Omega_{j0}} \right]^{\frac{1}{2}} \tag{49}$$

Thus, high values of $\bar{S}_{ij}$ indicate high importance of combination instability region with respect to the main one.

Figure 3 shows the variation of $\bar{S}_{12}$ vs the length-to-thickness ratio for the above HSDT and for the first-order shear deformation theory (FSDT), where the shear correction factor is 5/6[6]. The difference between the two theories is minor and they both indicate that $\bar{S}_{12}$ increases as a/h decreases. Figure 4 shows the variation of $\bar{S}_{12}$ vs the E_1/E_2 ratio, where $a/h = 4$. It is seen that $\bar{S}_{12}$ is increasing as E_1/E_2 is increasing up to 20 and then remains almost constant.

It should be noted that when symmetric cross-ply laminates are analyzed within the classical lamination theory, there is only one equation of motion, and thus a combination resonance, for each mode, is impossible.

Figure 5 shows the variation of $\bar{S}_{1j}$ (j = 2, 3, 4, 5) vs a/h for angle-ply laminates, $\theta = 5$ deg. It is clear that for thick plates $\bar{S}_{13}$ is the most important one. Yet, as the plates become thinner, $\bar{S}_{13}$ is the less important, whereas all the other ones are of the same importance. It is interesting to note that at $a/h = 7$ all $\bar{S}_{ij}$ are equal. Figure 6 shows the variation of $\bar{S}_{13}$ vs a/h for both HSDT and FSDT, where $\theta = 5$ deg, while Fig. 7 shows the variation of $\bar{S}_{13}$ vs θ where $a/h = 4$. In both figures it is seen that the contribution of the combination resonance is higher within the HSDT than the FSDT. At lower values of θ, $\bar{S}_{13}$ is most significant, whereas at $\theta = 45$ deg, $\bar{S}_{13} = 0$. Thus, from the combination resonance point of view, 45 deg is preferable for design.

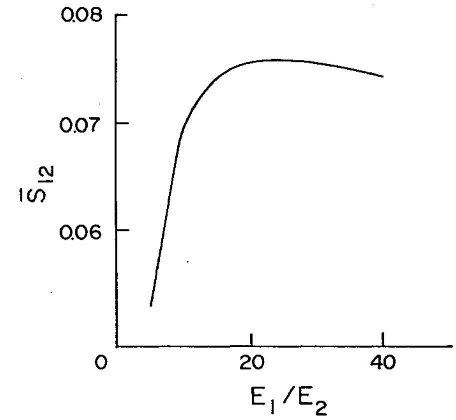

Fig. 4 Variation of $\bar{S}_{12}$ vs E_1/E_2 for symmetric cross-ply laminates, $a/h = 4$.

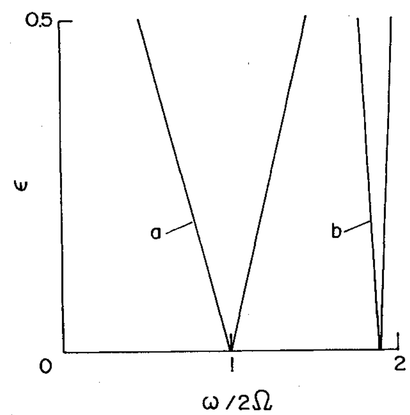

Fig. 2 Two instability regions of symmetric cross-ply laminate: a) $\omega = 2\Omega_1$; and b) $\omega = \Omega_1 + \Omega_2$.

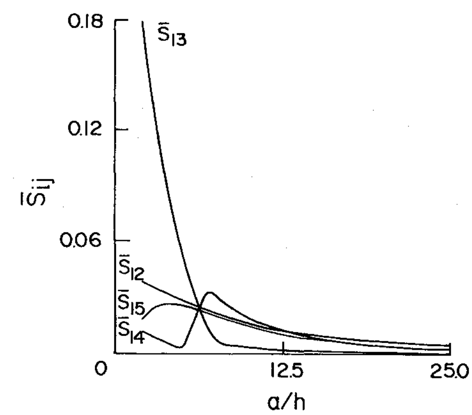

Fig. 5 Variation of $\bar{S}_{1j}$ vs a/h for antisymmetric angle-ply laminates, $\theta = 5$ deg.

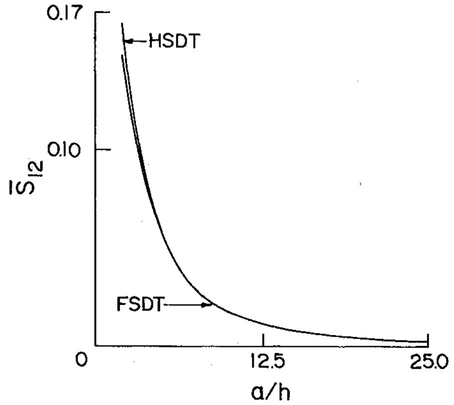

Fig. 3 Variation of $\bar{S}_{12}$ vs the length-to-thickness ratio for symmetric cross-ply laminates.

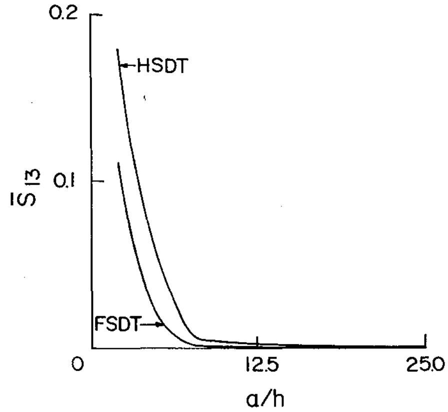

Fig. 6 Variation of $\bar{S}_{13}$ vs a/h for antisymmetric angle-ply laminates, $\theta = 5$ deg.

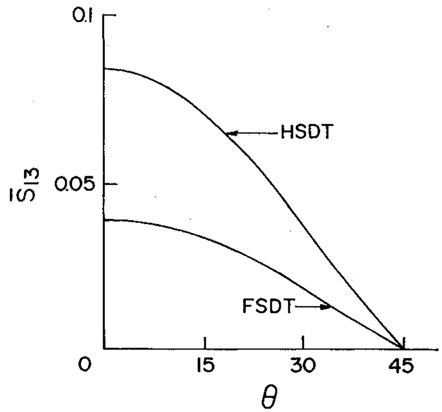

Fig. 7 Variation of $\bar{S}_{13}$ vs θ for antisymmetric angle-ply laminates, a/h = 4.

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